September 12, 2017
Class Notes

$\boxed{\text{LAST CLASS}}$

A language is regular iff an NFA recognizes it

1) Closure of regular languages under regular operations

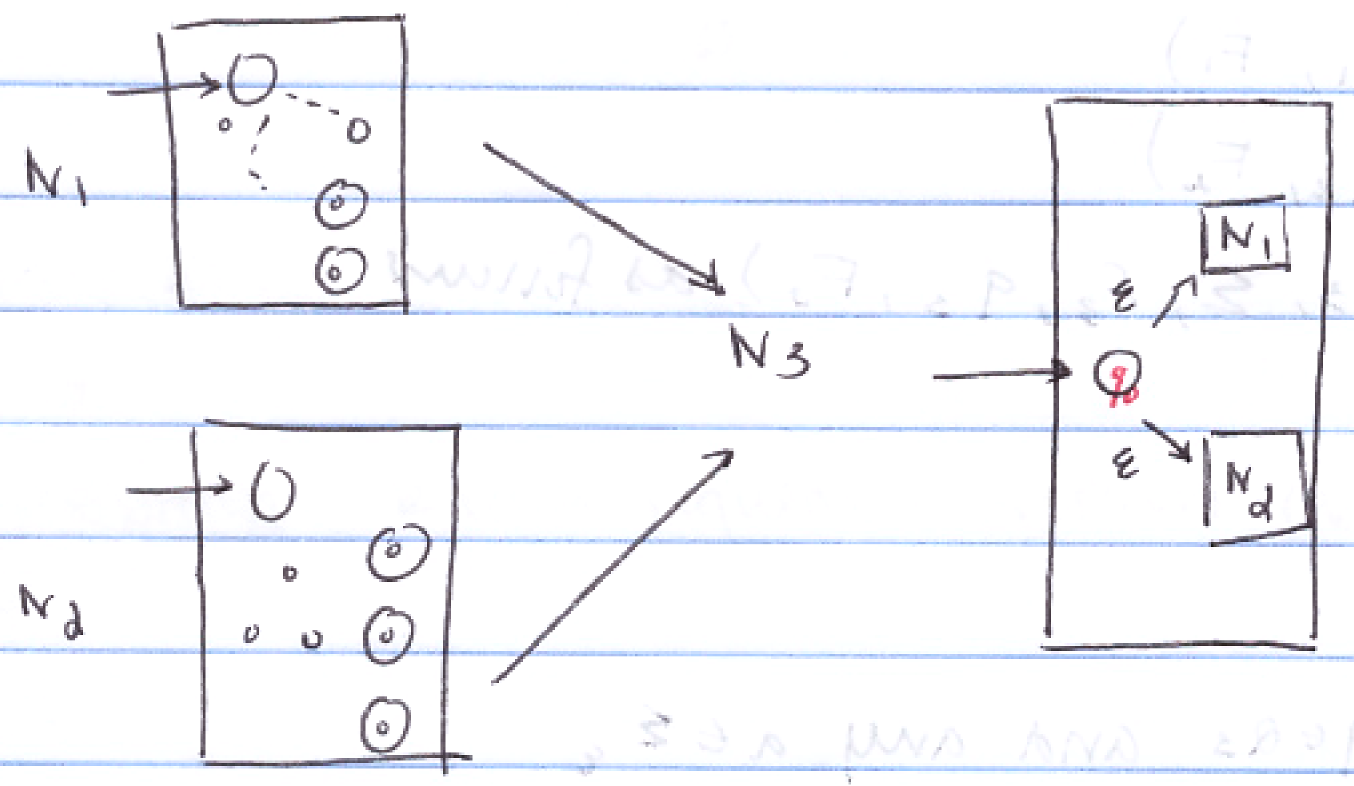
RECALL: union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

concatenation: $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$

star: $A^* \ \{x_1 \dots x_k \mid k \geq 0 \text{ and } x_i \in A \ \forall\, 1 \leq i \leq k\}$

Thm 1.25 Let A & B be regular languages. Then $A \cup B$ is regular, i.e. regular languages "closed under union"

Pf/ Idea: Let N_1 & N_2 be NFAs for A & B, respectively



Formally: Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$

$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$

We construct

$N_3 = (Q_3, \Sigma, \delta_3, q_0, F_3)$

as follows:

(1) $Q_3 = a \cup Q_2 \cup \{q_0\}$

(2) $q_3 = q_0$

(3) $F_3 = F_1 \cup F_2$

(4) Define δ_3 such that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$:

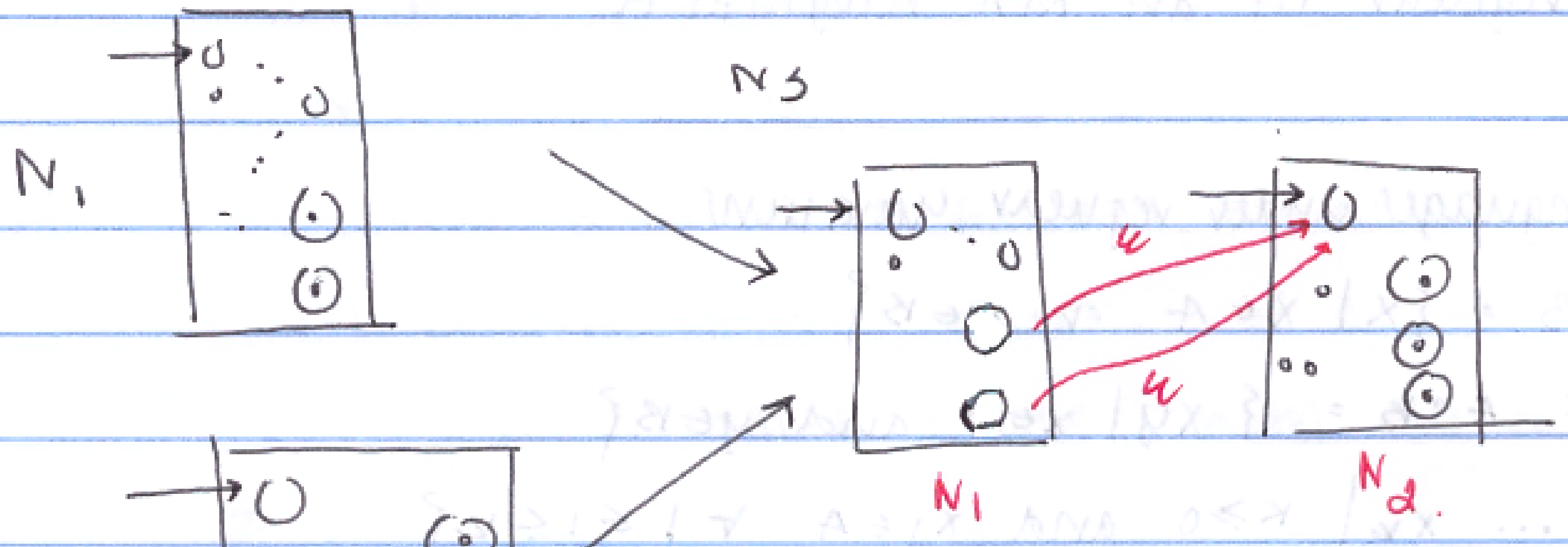
$$\delta_3(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1 \\ \delta_2(q, a) & q \in Q_2 \\ \{q_1, q_2\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$

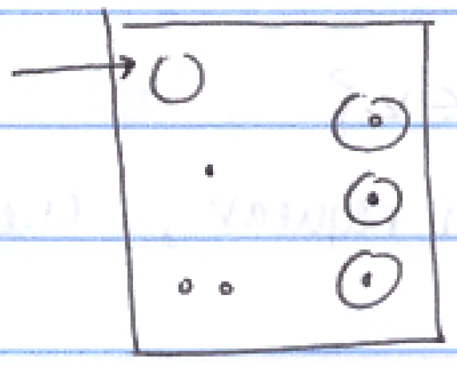
Thm 1.47 Let A & B be regular languages. Then $A \circ B$ is regular

i.e. regular languages "closed under concatenation"

$A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$

Pf/ Idea: Let N_1 & N_2 be NFAs for A & B, respectively pf / continued





e.g. $A = \{cat\}$ $\quad A \circ B = \{catbub\}$
$B = \{bub\}$

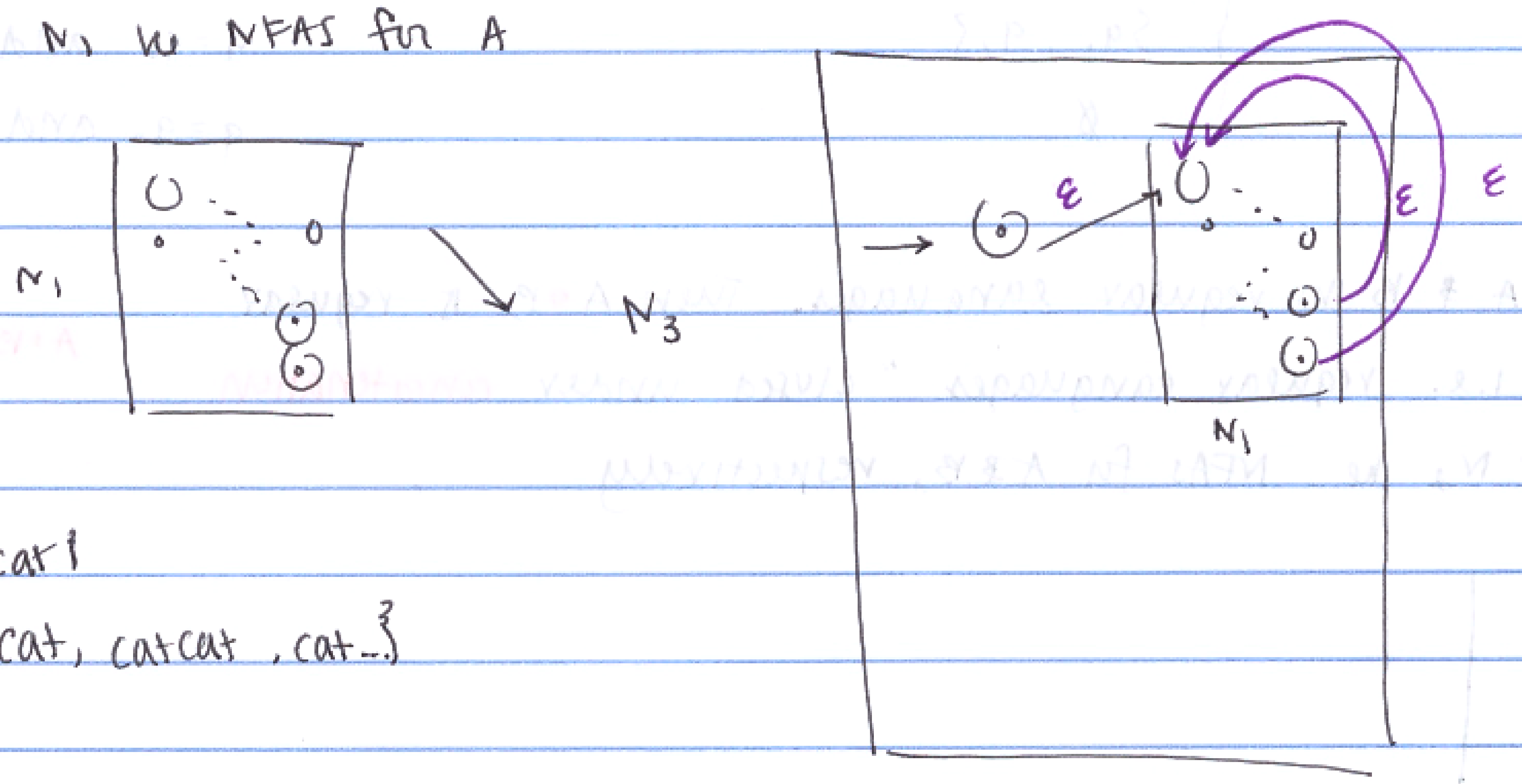
Formally: let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$
we construct $N_3 = (Q_3, \Sigma, \delta_3, q_3, F_3)$ as follows

(1) $Q_3 = Q_1 \cup Q_2$
(2) $q_3 = q_1$
(3) $F_3 = F_2$
(4) Define δ_3 s.t. For any $q \in Q_3$ and any $a \in \Sigma_\varepsilon$

$$\delta_3 = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_2(q,a) & q \in Q_2 \\ \delta_1(q_1,a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \{q_2\} \cup (q,a) & q \in F_1 \text{ and } a = \varepsilon \end{cases}$$

Thm 1.49 Let A be a regular language. Then A^* is regular
i.e. regular languages " closed under star " $\quad A^* = \{x_1 \dots x_k \mid k \geq 0, x_i \in A\}$

pf/ Idea: let N_1 be NFAs for A



e.g $A = \{cat\}$
$A^* = \{\varepsilon, cat, catcat, cat\dots\}$

Formally: let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$

we construct $N_3 = \{Q_3, \Sigma, \delta_3, q_3, F_3\}$ as follows

① $Q_3 = Q \cup \{q_0\}$

② $q_3 = q_0$

③ $F_3 = F_1 \cup \{q_0\}$

④ Define δ_3 s.t. for any $q \in Q_3$ and any $a \in \Sigma_\varepsilon$

$$\delta_3 = \begin{cases} \delta_1(q,a) & q \in Q \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q,a) \cup \{q_1\} & q \in F_1 \text{ and } a = \varepsilon \\ \{q_1\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$

2) Regular expressions

↳ Rich *

Formal definition: R is a regular expression (RE) if R is either of:

base cases:

① ε

② $a \in \Sigma$

③ $\emptyset$

④ $R_1 \cup R_2$ for RE's R_1 & R_2

⑤ $R_1 \circ R_2$ for RE's R_1 & R_2

⑥ R_1^* for RE R_1

Note: ① $R = \varepsilon \Rightarrow L(R) = \{\varepsilon\}$

② $R = a \Rightarrow L(R) = \{a\}$

③ $R = \emptyset \Rightarrow L(R) = \{\emptyset\}$

④ $R = R_1 \cup R_2 \Rightarrow L(R_1 \cup R_2) = L(R_1) \cup L(R_2)$

⑤ $R = R_1 \circ R_2 \Rightarrow L(R_1 \circ R_2) = L(R_1) \circ L(R_2)$

⑥ $R = R_1^* \Rightarrow L(R_1^*) = L(R_1)^*$

Precedence: star, then concatenation, then union (unless parentheses specify the order)

Define $R^+ := R' R^*$

example let $\Sigma = \{0,1\}$

1. $0^*10^* = \{w \mid w$ contains a single $1\}$
2. $\Sigma^* 1 \Sigma^* = \{w \mid w$ contains at least one $1\}$
3. $1^*(01^+)^* = \{w \mid$ every 0 in w is followed by at least one $1\}$
4. $(\Sigma\Sigma\Sigma)^* = \{w \mid$ length of w is multiple of 3.
5. $0 \cup 1 \cup 0\Sigma^*0 \cup 1\Sigma^*1 = \{w \mid w$ starts & ends with the same symbol$\}$